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Soliton stability in systems of two real scalar fields

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Abstract. In this paper we consider a class of systems of two coupled real scalar fields in bidimensional spacetime, the main motivation being the study of the classical or linear stability of soliton solutions. First, we present the class of systems and comment on the topological profile of soliton solutions one can find from the first-order equations that solve the equations of motion. We then follow the standard approach to classical stability to introduce the main steps one needs to obtain the spectra of Schrödinger operators that appear in this class of systems. We consider a specific system, from which we illustrate the general calculations and present some analytical results. We also consider another, more general, system and present an investigation that introduces new results and offers a comparison with former investigations.

1. Introduction

Lagrangian systems described by coupled scalar fields are recently gaining renewed attention. In the case of real fields, in particular, the work presented in [1–4] has introduced a specific class of systems of two coupled real scalar fields. In contrast to older models, the class of systems introduced in the above papers is very peculiar, at least in bidimensional spacetime, where it has the following general properties: first, the corresponding equations of motion are solved by field configurations obeying first-order differential equations; second, the classical configurations that solve the first-order equations have a minimum energy and are classically or linearly stable. On the other hand, the first-order differential equations can be seen as a dynamical system, and so we can take advantage of all the mathematical tools available to dynamical systems to deal with the set of first-order equations and, consequently, with solutions to the equations of motion.

For the specific issue concerning classical or linear stability, in [2, 4] a general way of investigating stability was presented. In this case, the investigation relies essentially on proving that the associate Schrödinger operator is positive semi-definite. This procedure of implementing stability investigations [5] is distinct from the so-called standard approach [5, 6], where one usually obtains the complete set of eigenvalues of the Schrödinger operator.

However, we know that the standard approach is important since it sets forward results one needs to implement quantum corrections, since in this case one has to know explicitly the spectrum of the corresponding Schrödinger operator [5, 6]. We therefore think that a standard investigation of classical stability of systems of two real scalar fields belonging to the class of systems already introduced in [1–4] is welcome, not only to present comparison with former investigations but also to unveil the full spectra of the corresponding Schrödinger operators. Within this context, the main motivation of this paper is to deal with issues concerning the finding of spectra of Schrödinger operators that naturally appear in the

standard approach to the linear stability of systems belonging to the class of coupled fields introduced in [1–4]. As we shall show, in this class of systems the Schrödinger operators can be written in terms of first-order operators, and this property eases the calculation toward obtaining the corresponding energy spectra.

Evidently, to implement the standard approach to linear stability we have to deal with specific systems. However, to make this investigation as general as possible, we have organized the work as follows. In section 2 we present the class of systems of two coupled real scalar fields and comment on the topological profile of the classical configurations. In section 3 we investigate linear stability within the standard approach, and shed new light on the issue concerning the unveiling of the full spectra of Schrödinger operators that appear in the class of systems introduced in the former section. In order to illustrate the general calculations, in section 4 we examine a particular system of two coupled fields. We end this paper in section 5, where we present another investigation, in which we deal with a more general system, not belonging to the class of systems introduced in section 2. Each section contains some comments and conclusions.

2. Systems of two real scalar fields

Let us start with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi - U \quad (1)$$

where $U = U(\phi, \chi)$ is the potential, in general a nonlinear function of the two fields ϕ and χ . Here we are using natural units, and so $\hbar = c = 1$, and the metric is such that $x^\alpha = (t, x)$ and $x_\alpha = (t, -x)$. The class of systems introduced in [1–4] is defined by the following potential

$$U(\phi, \chi) = \frac{1}{2} H_\phi^2 + \frac{1}{2} H_\chi^2 \quad (2)$$

where $H = H(\phi, \chi)$ is a smooth but otherwise arbitrary function of the fields ϕ and χ , and $H_\phi = \partial H / \partial \phi$, $H_\chi = \partial H / \partial \chi$.

The Euler–Lagrange equations, that is, the equations of motion that follow from the above system are given by

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + H_\phi H_{\phi\phi} + H_\chi H_{\phi\chi} = 0 \quad (3)$$

and

$$\frac{\partial^2 \chi}{\partial t^2} - \frac{\partial^2 \chi}{\partial x^2} + H_\phi H_{\chi\phi} + H_\chi H_{\chi\chi} = 0 \quad (4)$$

and for static field configurations they change to

$$\frac{d^2 \phi}{dx^2} = H_\phi H_{\phi\phi} + H_\chi H_{\phi\chi} \quad (5)$$

and

$$\frac{d^2 \chi}{dx^2} = H_\phi H_{\chi\phi} + H_\chi H_{\chi\chi}. \quad (6)$$

Recall that static field configurations are configurations written in their rest reference frame.

For static field configurations, the energy can be cast to the form $E = E_M + E'$, where

$$E_M = H(\phi(\infty), \chi(\infty)) - H(\phi(-\infty), \chi(-\infty)) \quad (7)$$

and

$$E' = \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\frac{d\phi}{dx} - H_\phi \right)^2 + \left(\frac{d\chi}{dx} - H_\chi \right)^2 \right]. \quad (8)$$

As we have already learned from [4], we impose the conditions

$$\frac{d\phi}{dx} = H_\phi \quad \frac{d\chi}{dx} = H_\chi \quad (9)$$

and in this case we see that the energy reaches its lower bound E_M , and the above first-order equations (9) solve the corresponding equations of motion (5) and (6).

Before investigating classical stability, let us first comment on the issue concerning topological properties of soliton solutions. From the above calculations on the energy of the corresponding static fields, we see that field configurations obeying the pair of first-order equations have a minimum energy that depends only on the difference between the two asymptotic behaviours of $H(\phi, \chi)$. In this case, classical pairs of static field configurations having finite but nonzero energy must necessarily obey the topological property of connecting distinct minima of the corresponding potential. These types of field configurations are called topological solutions [6]. However, we know that systems of two coupled scalar fields may have nontopological solutions [6], and in this case the field configurations must have the same asymptotic behaviour. In the above class of systems, however, static field configurations obeying the first-order differential equations (9) cannot have nontopological profiles, because this would give zero energy to the pair of solutions, and zero is the energy value of the vacuum states. This result is very interesting, because it leads us to the fact that the search for classical solutions of the equations of motion via first-order equations (9) evidently does not give us the full set of physical solutions, and certainly no nontopological solutions.

Another interesting point comes from considering the set of first-order equations as a dynamical system. This procedure allows us to unveil the full set of singular points, together with the stability properties of each one of these points. As we can see from the potential (2), the set of singular points is identified with the vacuum manifold of the corresponding field theory. Then, once we know the vacuum manifold and the classification of each one of its points as stable, unstable and saddle points, we have everything we need to deal with finding explicit soliton solutions. The route toward finding explicit solutions can, for instance, follow the trial-orbit method introduced in [7]. However, since here we are dealing with first-order equations this trial-orbit method becomes easier to implement than in the original work [7].

3. Classical stability

Let us now focus attention on the issue concerning classical or linear stability. In this case we consider $\bar{\phi} = \bar{\phi}(x)$ and $\bar{\chi} = \bar{\chi}(x)$ as a pair of static solutions to the above first-order differential equations. Here we consider $\phi(x, t) = \bar{\phi}(x) + \eta_n(x) \cos(w_n t)$ and $\chi(x, t) = \bar{\chi}(x) + \xi_n(x) \cos(w_n t)$ in order to get from the equations of motion (3) and (4) and to work up to first order in the fluctuations,

$$S_2 \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = w_n^2 \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}. \quad (10)$$

This is a Schrödinger equation, and S_2 is the Schrödinger operator, given by

$$S_2 = -\frac{d^2}{dx^2} + V \quad (11)$$

where the potential V has the form

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \quad (12)$$

and the matrix elements can be written as

$$V_{11} = \bar{H}_{\phi\phi}^2 + \bar{H}_{\chi\phi}^2 + \bar{H}_\phi \bar{H}_{\phi\phi\phi} + \bar{H}_\chi \bar{H}_{\phi\phi\chi} \quad (13)$$

and

$$V_{12} = V_{21} = \bar{H}_{\phi\phi} \bar{H}_{\phi\chi} + \bar{H}_{\chi\chi} \bar{H}_{\phi\chi} + \bar{H}_\phi \bar{H}_{\phi\phi\chi} + \bar{H}_\chi \bar{H}_{\phi\chi\chi} \quad (14)$$

and

$$V_{22} = \bar{H}_{\chi\chi}^2 + \bar{H}_{\phi\chi}^2 + \bar{H}_\phi \bar{H}_{\phi\chi\chi} + \bar{H}_\chi \bar{H}_{\chi\chi\chi}. \quad (15)$$

In the above expressions a bar over H means that the corresponding quantity has to be calculated at the classical static values $\phi = \bar{\phi}(x)$ and $\chi = \bar{\chi}(x)$, and so $V_{ij} = V_{ij}(x)$, for $i, j = 1, 2$.

On the other hand, as was already shown in [2, 4], from the first-order equations we can introduce the first-order operators

$$S_1^\pm = \pm \frac{d}{dx} + v \quad (16)$$

where v is given by

$$v = \begin{pmatrix} H_{\phi\phi} & H_{\phi\chi} \\ H_{\phi\chi} & H_{\chi\chi} \end{pmatrix}. \quad (17)$$

Here it is not difficult to check that these first-order operators are adjoint of each other, and that $S_2 = S_1^+ S_1^-$. This is important because it proves that the Schrödinger operator S_2 is positive semi-definite, and this is the result one needs to ascertain that the pair of solutions $\bar{\phi}$ and $\bar{\chi}$ is classically or linearly stable.

To investigate the spectrum of the Schrödinger operator S_2 we have to go further into this problem, and here the main difficulty concerns solving the Schrödinger equation (10). The task is not immediate since in the case of two coupled fields the fluctuations η and ξ are also coupled, in general, and so the first issue we have to deal with concerns the diagonalization at the second-order Schrödinger operator, a calculation to be done by finding the normal mode fluctuations.

In the above class of systems of two coupled fields, however, we take advantage of the presence of the corresponding first-order operators S_1^\pm , and so the task of finding the normal mode fluctuations is greatly simplified. This is so because here we can deal with the simpler task of just diagonalizing the v matrix which appears in the first-order operators. In this case the result allows us to write

$$\bar{S}_1^\pm = \pm \frac{d}{dx} + \begin{pmatrix} v_+ & 0 \\ 0 & v_- \end{pmatrix} \quad (18)$$

where the diagonal elements are given by

$$v_\pm = \frac{1}{2}(\bar{H}_{\phi\phi} + \bar{H}_{\chi\chi}) \pm [\frac{1}{4}(\bar{H}_{\phi\phi} - \bar{H}_{\chi\chi})^2 + \bar{H}_{\phi\chi}^2]^{1/2}. \quad (19)$$

If we use the notation η_\pm for the pair of normal mode fluctuations, then we have to deal with the following Schrödinger equations

$$\bar{S}_2^\pm \eta_\pm = w^2 \eta_\pm \quad (20)$$

where the Schrödinger operators \bar{S}_2^\pm are now given by

$$\bar{S}_2^\pm = -\frac{d^2}{dx^2} + V_\pm \quad (21)$$

with the potentials

$$V_\pm = v_\pm^2 + \frac{dv_\pm}{dx}. \quad (22)$$

Before paying attention to specific systems, let us reason a little more on the issue concerning linear stability. As we can see from the above investigation, to unveil the spectra of \bar{S}_2^\pm we recognize that the square root that appears in v_\pm complicates the calculation, and will certainly require numerical investigations. To circumvent this difficulty and perhaps give explicit analytical results, we should focus attention on avoiding the square root in v_\pm . Here we see that the simplest case where the square root in v_\pm disappears is when $\bar{H}_{\phi\chi} = 0$. However, since we are dealing with systems of two coupled scalar fields, to work with nontrivial systems we must have $H_{\phi\chi}$ nonzero, in order to account for interactions between the two fields. In this way, to get to $\bar{H}_{\phi\chi} = 0$, the pair of classical solutions $\bar{\phi}$ and $\bar{\chi}$ must be very specific. However, in general the quantity $\bar{H}_{\phi\chi}$ does contribute, and so we must first deal with the quantity

$$R = \frac{1}{4}(H_{\phi\phi} - H_{\chi\chi})^2 + H_{\phi\chi}^2 \quad (23)$$

which appears inside the square root in v_\pm . Here we should work out a way to prevent the classical pair $\bar{\phi}$ and $\bar{\chi}$ remaining inside the square root. This reasoning will become clearer in the following, where we deal with a specific system.

4. An example

As a particular example, let us consider the system defined by the function

$$H(\phi, \chi) = \lambda(\frac{1}{3}\phi^3 - a^2\phi) + \frac{1}{2}\mu\phi\chi^2. \quad (24)$$

In this case the potential that specifies the system has the form

$$U(\phi, \chi) = \frac{1}{2}\lambda^2(\phi^2 - a^2)^2 + \frac{1}{2}\lambda\mu(\phi^2 - a^2)\chi^2 + \frac{1}{8}\mu^2\chi^4 + \frac{1}{2}\mu^2\phi^2\chi^2. \quad (25)$$

Our main motivation to work with the above system is that it is similar to the model already investigated in [7], for which a standard stability investigation has already been carried out [8], and that has pairs of soliton solutions [1] that are very similar to the pairs considered in [8]. This motivation broadens with the fact that this specific system was shown to be useful not only in field theory [3] but also in condensed matter [4].

In this case the first-order equations become

$$\frac{d\phi}{dx} = \lambda(\phi^2 - a^2) + \frac{1}{2}\mu\chi^2 \quad (26)$$

and

$$\frac{d\chi}{dx} = \mu\phi\chi. \quad (27)$$

This system was already investigated in [1], and some pairs of soliton solutions were presented. In particular, a pair of solutions is

$$\bar{\phi}_1(x) = -a \tanh(\lambda ax) \quad \bar{\chi}_1(x) = 0. \quad (28)$$



Figure 1. The two pairs of soliton solutions.

We recall that this pair of solutions introduces no restrictions on the two parameters λ and μ . Another pair of solutions is

$$\bar{\phi}_2(x) = -a \tanh(\mu ax) \quad (29)$$

$$\bar{\chi}_2(x) = \pm a \sqrt{2 \left(\frac{\lambda}{\mu} - 1 \right)} \operatorname{sech}(\mu ax). \quad (30)$$

For this second pair of solutions the parameters λ and μ are restricted to satisfy $\lambda/\mu > 1$. Note that the limit $\lambda/\mu \rightarrow 1$ transforms the second pair of solutions into the first one. Note also that for the second pair of solutions the field configurations obey

$$\bar{\phi}_2^2 + \frac{1}{2} \left(\frac{\lambda}{\mu} - 1 \right)^{-1} \bar{\chi}_2^2 = a^2. \quad (31)$$

We see that both pairs of solutions connect the points $(a, 0)$ and $(-a, 0)$ in the (ϕ, χ) plane, the first by a straight line, and the second by an elliptical line, as shown in figure 1.

As we can see, these two pairs of soliton solutions belong to the same topological sector, and have the same energy [1, 4]. Furthermore, the solutions are very similar to the pairs of classical configurations investigated in [8], and so it seems interesting to compare these calculations with the ones introduced there. Here we recall that we already know that the above pairs of solutions are stable [2, 4], while the pairs considered in [8] were shown to be unstable, at least in the region of parameters considered. This is an interesting result, since it shows that, in contrast to the class of systems defined by the function $H = H(\phi, \chi)$, older systems such as the one presented in [7] may have no classically stable soliton solutions, and this is true at least in some region of parameter space.

To obtain the spectra of the corresponding Schrödinger operators we use $H(\phi, \chi)$ given by (24) to write $H_{\phi\phi} = 2\lambda\phi$, $H_{\chi\chi} = \mu\phi$, and $H_{\phi\chi} = \mu\chi$. For the first pair of solutions we obtain $\bar{H}_{\phi\chi} = 0$, and so the equations for the fluctuations are already decoupled. This is very specific, and appears because of the classical value $\bar{\chi} = 0$. In this case we obtain

$$v_+ = -2\lambda a \tanh(\lambda ax) \quad (32)$$

and

$$v_- = -\mu a \tanh(\lambda ax). \quad (33)$$

For the second pair of solutions the fluctuations are coupled, and so first we consider R as given by equation (23). Here we have

$$R = \mu^2 \left(\frac{\lambda}{\mu} - \frac{1}{2} \right)^2 \bar{\phi}_2^2 + \mu^2 \bar{\chi}_2^2. \quad (34)$$

We use the orbit (31) to rewrite the above quantity as

$$R = \mu^2 \left(\frac{\lambda}{\mu} - \frac{3}{2} \right)^2 \bar{\phi}_2^2 + 2\mu^2 \left(\frac{\lambda}{\mu} - 1 \right) a^2. \quad (35)$$

In this case the parameters obey $\lambda/\mu > 1$, and so the only way of removing the classical field from the square root is by setting the coefficient of $\bar{\phi}$ to zero. Here we get the relation $\lambda = \frac{3}{2}\mu$. This is a very interesting point in parameter space since it equals the amplitude

of the two classical configurations, as we can immediately see from the second pair of solutions; also, the value $\lambda = \frac{3}{2}\mu$ changes the general elliptical profile of the orbit (31) to the very particular case of a circular one.

For the first pair of solutions, the Schrödinger operators corresponding to fluctuations about the ϕ and χ fields are given by, respectively,

$$\bar{S}_2^{(1,1)} = -\frac{d^2}{dx^2} + 4\lambda^2 a^2 - 6\lambda^2 a^2 \operatorname{sech}^2(\lambda ax) \quad (36)$$

and

$$\bar{S}_2^{(1,2)} = -\frac{d^2}{dx^2} + \mu^2 a^2 - \mu(\lambda + \mu)a^2 \operatorname{sech}^2(\lambda ax). \quad (37)$$

This system has already been investigated in [1]. From the results obtained there we see that, for $\lambda = \frac{3}{2}\mu$, fluctuations about the ϕ field include the zero mode, and a bound state at the value $w^2 = \frac{3}{4}9\mu^2 a^2$, with the continuum starting at $9\mu^2 a^2$. For fluctuations about the χ field, only the zero mode is present, and the continuum starts at the value $\mu^2 a^2$.

For the second pair of solutions, the Schrödinger operators corresponding to fluctuations about the normal modes can be written as

$$\bar{S}_2^{(2,\pm)} = -\frac{d^2}{dx^2} + 5\mu^2 a^2 \mp 4\mu^2 a^2 \tanh(\mu ax) - 6\mu^2 a^2 \operatorname{sech}^2(\mu ax). \quad (38)$$

In this case we see [9] that these fluctuations have zero modes and no other bound states. Furthermore, the continua start at $\mu^2 a^2$, and are formed by reflecting states in the interval $\mu^2 a^2$ and $9\mu^2 a^2$, and free states for energies greater than $9\mu^2 a^2$.

Before ending this section, we note that fluctuations about the first pair of solutions are described by reflectionless potentials, for which the continua start at $9\mu^2 a^2$ and $\mu^2 a^2$, as shown in figure 2.

On the other hand, fluctuations about the second pair of solutions are described by potentials that accommodate reflecting states, and these reflecting states appear between the values $\mu^2 a^2$ and $9\mu^2 a^2$, as depicted in figure 3.

This property is in fact independent of the particular ratio between the two parameters: for $\lambda/\mu > 1$, it is not hard to show that fluctuations about the second pair of solutions are always described by potentials that accommodate the reflecting states between $\mu^2 a^2$ and $4\lambda^2 a^2$, which are exactly the values where the continua of the reflectionless potentials that describe fluctuations about the first pair of solutions start.

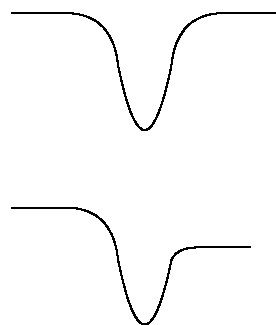


Figure 2. Potentials for the first pair of solutions.

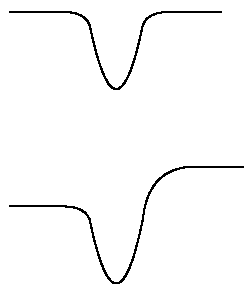


Figure 3. Potentials for the second pair of solutions.

5. A general system

Let us now consider another system, defined by the potential

$$U(\phi, \chi) = \frac{1}{4}(\phi^2 - 1)^2 + \frac{1}{2}f\chi^2 + \frac{1}{4}\lambda\chi^4 + \frac{1}{2}d(\phi^2 - 1)\chi^2. \quad (39)$$

This is the potential considered in [8], and here we use the same notation, with λ , f and d real and positive. In this case we see that

$$U(\phi, 0) = \frac{1}{4}(\phi^2 - 1)^2 \quad (40)$$

and so with this system we obtain the pair of solutions

$$\bar{\phi}_3(x) = \tanh(x/\sqrt{2}) \quad \bar{\chi}_3 = 0. \quad (41)$$

In [8] it was shown that this pair is unstable, at least in some region of parameter space. We recall that the investigation carried out in [8] was mainly concerned with the stability of another pair of solutions, of the same form as the one given by the second pair of solutions in the former example. In that investigation, it was also shown [8] that when the parameters of the system allow for a normal mode diagonalization that leads to analytical results, the pair of solutions given by the above equation (41) is classically unstable.

Our main interest here is simpler, and concerns the investigation of whether there is some region in parameter space where the above pair of solutions is classically or linearly stable. In this case, the Schrödinger operators corresponding to small fluctuations $\eta_n(x)$ and $\xi_n(x)$ described by $\phi(x, t) = \bar{\phi}_3 + \eta_n(x) \cos(w_n t)$ and $\chi(x, t) = \bar{\chi}_3 + \xi_n(x) \cos(w_n t)$ can be cast to the following forms

$$\bar{S}_2^{(3,1)} = -\frac{d^2}{dx^2} + 2 - 3 \operatorname{sech}^2(x/\sqrt{2}) \quad (42)$$

$$\bar{S}_2^{(3,2)} = -\frac{d^2}{dx^2} + f - d \operatorname{sech}^2(x/\sqrt{2}). \quad (43)$$

The first operator includes two bound states: the zero mode and another bound state at $w^2 = \frac{3}{2}$. However, for the second operator we see that a set of $n = 0, 1, 2, \dots$ bound states can appear, where n obeys

$$n < \frac{1}{2} \left[\sqrt{1 + 8d} - 1 \right]. \quad (44)$$

This shows that the number of bound states depends only on the parameter d , and that there is at least one bound state for $d > 0$. The energy of the bound states are given by

$$w_n^2 = f - \frac{1}{8} \left[\sqrt{1 + 8d} - 1 - 2n \right]^2. \quad (45)$$

To avoid instability we focus attention on the deepest bound state: here we see that to ensure stability we have to impose the restriction

$$f \geq \frac{1}{4} \left[1 + 4d - \sqrt{1 + 8d} \right] \quad (46)$$

and this shows that there is room to choose d and f , keeping the corresponding pair of solutions stable. Furthermore, we remark that since $1 + 4d$ is always greater than $\sqrt{1 + 8d}$ for $d > 0$, one cannot set $f \rightarrow 0$ because this would unavoidably introduce instability.

On the other hand, stability does not impose any restriction on the sign of $d - f$, and this allows us to go a little further on this issue since for $d - f > 0$ the above potential has other minima. To see this explicitly, let us note that the potential (39) also gives

$$U(0, \chi) = \frac{1}{4} - \frac{1}{2}(d - f)\chi^2 + \frac{1}{4}\lambda\chi^4. \quad (47)$$

Here we see that for $d - f \leq 0$ the points $(\phi_0^2 = 1, \chi = 0)$ are the only possible global minima of the potential. However, for $d - f > 0$ there are other minima, at $\phi = 0$ and

$$\chi_0^2 = \frac{d - f}{\lambda} \quad (48)$$

and these minima can be local or global minima, depending on the value of the other parameter, λ . For simplicity, let us suppose that $\lambda \geq (d - f)^2$. In this case the points $(\pm 1, 0)$ are always global minima, but when $\lambda = (d - f)^2$ there are also global minima at $(0, \pm\sqrt{1/(d - f)})$.

For $d - f = r > 0$, and for $\lambda \geq r^2$ we can get another pair of solutions, of the same type as the former one, given by

$$\bar{\phi}_4 = 0 \quad \bar{\chi}_4(x) = \sqrt{\frac{r}{\lambda}} \tanh\left(\sqrt{r/2}x\right). \quad (49)$$

To investigate stability we proceed as before: Here the Schrödinger operators are

$$\bar{S}_2^{(4,1)} = -\frac{d^2}{dx^2} + \left[\frac{d}{\lambda}r - 1\right] - \frac{d}{\lambda}r \operatorname{sech}^2\left(\sqrt{r/2}x\right) \quad (50)$$

$$\bar{S}_2^{(4,2)} = -\frac{d^2}{dx^2} + 2r - 3r \operatorname{sech}^2\left(\sqrt{r/2}x\right). \quad (51)$$

We see that $\bar{S}_2^{(4,2)}$ is like $\bar{S}_2^{(3,1)}$, that is, it has zero mode and another bound state at $w^2 = \frac{3}{2}r$. However, for the other Schrödinger operator the number of bound states is now controlled by

$$n < \frac{1}{2} \left[\sqrt{1 + 8\frac{d}{\lambda}} - 1 \right] \quad (52)$$

and so there is at least one bound state. From the energy of the deepest bound state we obtain, to ensure stability of the corresponding pair of solutions,

$$\frac{d}{\lambda} - \frac{1}{r} \geq \frac{1}{4} \left[1 + 4\frac{d}{\lambda} - \sqrt{1 + 8\frac{d}{\lambda}} \right]. \quad (53)$$

This restriction implies that $\lambda < dr$. Now, if we write $\lambda = sr^2$ we obtain

$$1 \leq s < \frac{d}{r} = \frac{d}{d - f} \quad (54)$$

and this implies that there are many possibilities of choosing s without destroying the stability of the corresponding pair of solutions. In particular, we can choose $s = 1$ to write $\lambda = r^2 = (d - f)^2$, which shows that in this case the potential (39) presents four degenerate global minima: two at $\chi = 0$ and $\phi^2 = 1$, and two at $\phi = 0$ and $\chi^2 = 1/r = 1/(d - f)$.

The above results show that when the system presents global minima at the four points $(\pm 1, 0)$ and $(0, \pm 1/\sqrt{r})$ we can have stable solitons joining the minima $(\pm 1, 0)$ by a straight line with $\chi = 0$ or the minima $(0, \pm 1/\sqrt{r})$ by another straight line with $\phi = 0$. This is interesting, at least within the context of searching for defects inside defects, as recently considered in [10], in the case of systems of the type considered in this section, and in [3], in the case of systems belonging to the class of systems introduced in section 2. However, this is another issue, which is currently under consideration.

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